

WEIERSTRASS CYCLES IN MODULI SPACES AND THE KRICHEVER MAP

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ABSTRACT. We analyze cohomological properties of the Krichever map and use the results to study Weierstrass cycles in moduli spaces and the tautological ring.

1. INTRODUCTION

Let us consider a point p on a smooth compact complex curve C having genus g . We say that a natural number n is a non-gap if there exists a function that is holomorphic on $C \setminus p$ and has a pole of order n at the point p (in other words $h^0(\mathcal{O}(np)) \neq 0$). We denote the set of all non-gaps by H (or by H_p if it is necessary to emphasize that we consider the non-gaps at the point p). It is obvious that H is a semigroup; it is easy to derive from Riemann-Roch theorem that the number of gaps (the number of elements of $\mathbb{N} \setminus H$) is equal to g . One says that H is the Weierstrass semigroup at p .

We say that a subsemigroup H of \mathbb{N} such that $\#(\mathbb{N} \setminus H) = g$ is a numerical semigroup of genus g ; obviously any Weierstrass semigroup belongs to this class.

One says that p is a Weierstrass point if the first non-gap is $\leq g$ (i.e. $H \neq \{g+1, g+2, \dots\}$). There exist only finite number of Weierstrass points on a curve.

Instead of Weierstrass semigroup H , one can consider a decreasing sequence of integers such that s_k is the largest integer with

$$h^0(K_C(-s_k p)) = k.$$

Here K_C denotes the canonical line bundle. It follows from Riemann-Roch theorem that this sequence (the Weierstrass sequence of the point p) has the form $s_1 = a_g - 1, \dots, s_g = a_1 - 1 = 0, s_{g+1} = -2, \dots, s_{g+k} = -(k+1), \dots$. Here a_1, \dots, a_g denotes the increasing sequence of gaps.

Notice that all these statements remain correct if p is a non-singular point of irreducible Cohen-Macaulay curve and the canonical line bundle is replaced with the dualizing sheaf ω_C .

Let us consider the moduli space of $\mathcal{M}_{g,1}$ of non-singular irreducible curves of genus g with one marked point (one can characterize this space as the universal curve). If H is numerical semigroup of genus g , we denote by \mathcal{M}_H the subset of $\mathcal{M}_{g,1}$ consisting of curves with marked points having Weierstrass semigroup H . The closure $W_H = \overline{\mathcal{M}_H}$ of the Weierstrass set \mathcal{M}_H in $\mathcal{M}_{g,1}$ is called a Weierstrass cycle. Under some conditions, we calculate the cohomology class $[W_H]$ dual to this cycle (our methods can be used also to calculate the element of Chow ring specified by Weierstrass cycle).

Our problem is closely related to the problem of the calculation of the homomorphism induced by the Krichever map $k : \widehat{\mathcal{M}}_g \rightarrow \text{Gr}(\mathcal{H})$. Here $\widehat{\mathcal{M}}_g$ stands for the moduli space of triples (C, p, z) , where C is a compact complex connected curve of genus g with a point p and a map $z : D \rightarrow \mathbb{D}$ is an isomorphism from a closed set D into the closed unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ obeying $z(p) = 0$. We use the notation $\text{Gr}(\mathcal{H})$ for the Sato

Grassmannian. The Krichever map sends a triple (C, p, z) into the space V of functions on the boundary of the disk D that can be extended to holomorphic differentials on the complement of D . (A function $f(z)$ on S^1 is considered as a differential $f(z)dz$ restricting to the boundary of D .) The kernel and the cokernel of $\pi_-|_V : V \rightarrow \mathcal{H}_-$ are identified with $H^0(C, \omega_C)$ and $H^1(C, \omega_C)$ respectively (see [14], [18]). Here π_- stands for the orthogonal projection of V into \mathcal{H}_- ; the projection is defined with respect to Hermitian inner product $\langle f_1, f_2 \rangle = \int_{S^1} f_1(z) \overline{f_2(z)} dz / 2\pi$. It follows from the Riemann-Roch theorem that π_- has index $g - 1$. Hence the image of the Krichever map lies in the component $\text{Gr}_{g-1}(\mathcal{H})$. The Krichever map commutes with the natural action of S^1 on $\widehat{\mathcal{M}}_g$ and on $\text{Gr}(\mathcal{H})$. Thus it induces a homomorphism of the equivariant cohomology of the connected component of $\text{Gr}(\mathcal{H})$ into equivariant cohomology of $\widehat{\mathcal{M}}_g$. The latter is isomorphic to the conventional cohomology of $\mathcal{M}_{g,1}$ (see [8] for more detail). In [8], we have calculated the homomorphism induced by the Krichever map on multiplicative generators of equivariant cohomology of Grassmannian; in the present paper we will give an explicit formula for this homomorphism on additive generators of this cohomology. In the paper [9], we identified the equivariant cohomology of Grassmannian with the ring of shifted symmetric functions. We describe the homomorphism induced by the Krichever map on this ring; we specify the answers for various additive generators of equivariant cohomology.

Weierstrass cycles W_H are related to intersections of Schubert cycles in the Grassmannian with Krichever locus (with the image of Krichever map). This allows us to obtain the information about classes $[W_H]$ from the analysis of homomorphism of cohomology induced by the Krichever map. The same techniques is used to obtain relations in the tautological rings of moduli spaces.

In a separate paper [10] we show how to use the ideas of present paper to obtain estimates for dimensions of Weierstrass cycles. We perform calculations for low genera.

2. KRICHEVER MAP

In the introduction, we have described the Krichever map $k : \widehat{\mathcal{M}}_g \rightarrow \text{Gr}(\mathcal{H})$ of the moduli space $\widehat{\mathcal{M}}_g$ into Segal-Wilson version of Sato Grassmannian (see [18] for more detail). This construction can be generalized to irreducible Cohen-Macaulay curves (the marked point p should be non-singular, instead of holomorphic differentials one should consider sections of the dualizing sheaf.) This follows from the results of [18] and from the remark that the dualizing sheaf of Cohen-Macaulay curve is a torsion-free rank one sheaf. We will denote the moduli space of triples (C, p, z) where C is an irreducible Cohen-Macaulay curve of genus g , p is a non-singular point and z is a coordinate on a disk centered at p by $\widehat{\mathcal{CM}}_g$; the extension of the Krichever map to this space will be also denoted by k . The extended Krichever map is an embedding of $\widehat{\mathcal{CM}}_g$ into Grassmannian; we can define the topology on $\widehat{\mathcal{CM}}_g$ using this embedding. The image of this embedding is called Krichever locus.¹

Notice that a reasonable moduli space of Cohen-Macaulay curves (even of Gorenstein curves) does not exist; see [7]. It is important that we consider curves with embedded disks. Identifying the points of $\widehat{\mathcal{CM}}_g$ corresponding to the same curve C with different embedded disks we obtain a non-separable space.

We have used the dualizing sheaf in the construction of Krichever map, however, as it was shown in [18], one can use any torsion-free rank one sheaf.

¹The Krichever map can be defined also for reducible curves, but in this case this map is not an embedding and it is not continuous. In particular, the Krichever map on the space of nodal curves with disks is discontinuous.

Using q -differentials one can construct more general Krichever map $k_q : \widehat{\mathcal{M}}_g \rightarrow \text{Gr}(\mathcal{H})$; this corresponds to using the q -th power of dualizing sheaf. Notice that $k_1 = k$. In general the map k_q for $q > 1$ cannot be defined for Cohen-Macaulay curves, but it can be defined for Gorenstein curves.

It is easy to check that the images of k_q and k_{1-q} are orthogonal with respect to bilinear inner product $(f_1, f_2) = \int_{S^1} f_1(z)f_2(z)dz/2\pi$; moreover,

$$(2.1) \quad k_{1-q}(C, p, z) = k_q(C, p, z)^\perp$$

where $^\perp$ denotes orthogonal complement (see [17]). In particular,

$$(2.2) \quad k_0(C, p, z) = k_1(C, p, z)^\perp.$$

One should emphasize that (2.2) is correct for all irreducible Cohen-Macaulay curves, but to prove (2.1) we should assume that C is a Gorenstein curve. All maps k_q are S^1 -equivariant; we study the induced homomorphisms on the equivariant cohomology. The answers are formulated in terms of lambda-classes and psi-classes.

The Hodge bundle \mathbb{E} on $\widehat{\mathcal{CM}}_g$ is defined as a bundle having the space of holomorphic sections of dualizing sheaf as a fiber. This is an equivariant vector bundle whose equivariant Chern classes are called lambda-classes and denoted by $\lambda_1, \dots, \lambda_g$. Restricting them to $\widehat{\mathcal{M}}_g$, we obtain conventional lambda-classes. (Recall, that the equivariant cohomology of $\widehat{\mathcal{M}}_g$ coincides with cohomology of universal curve $\mathcal{M}_{g,1}$, see [8].) Lambda-classes can be considered as elementary symmetric functions of lambda-roots (of Chern roots of the Hodge bundle).

S^1 -equivariant cohomology can be regarded as a module over polynomial ring $\mathbb{C}[u]$. The psi-class $\psi \in H_{S^1}(\widehat{\mathcal{CM}}_g)$ will be defined as $-u$. It was shown in [8] that restricting to $\widehat{\mathcal{M}}_g$ we obtain the standard definition of psi-class.

The subring of the ring $H_{S^1}(\widehat{\mathcal{CM}}_g)$ generated by lambda-classes and psi-class will be called tautological ring. It will follow from our results that the tautological ring can be characterized as the image of equivariant cohomology of Grassmannian by the homomorphism k^* induced by the Krichever map. We will prove some relations in the tautological ring; these relations can be restricted to relations in the tautological ring of the universal curve.

Let us consider submanifolds Gr_d^l of $\text{Gr}_d(\mathcal{H})$ consisting of points W such that the projection $\pi_l : W \rightarrow z^{-l}\mathcal{H}_-$ is surjective. (Here $l \geq 0$). The action of S^1 on $\text{Gr}_d(\mathcal{H})$ generates an action on Gr_d^l for each $l \geq 0$. The kernels of the projection $\pi_l : W \rightarrow z^{-l}\mathcal{H}_-$ can be considered as fibers of an equivariant vector bundle \mathcal{E}_l over Gr_d^l . This bundle has rank $d+l$.

Using the Krichever map we can embed $\widehat{\mathcal{CM}}_g$ into Gr_{g-1}^1 ; the Hodge bundle is a pull-back of the equivariant vector bundle \mathcal{E}_1 . (This statement can be considered as a rigorous definition of the Hodge bundle.)

We have proved in [9] that the S^1 -equivariant cohomology ring of Grassmannian $\text{Gr}_d(\mathcal{H})$ can be identified with the ring of polynomial functions of infinite number of variables $(x_i)_{i \in \mathbb{N}}$ and variable u that become symmetric with respect x_i after the shift of variables $x_i \rightarrow x_i + (d+1-i)u$. Equivalently, these functions (shifted symmetric functions of [1]) can be regarded as symmetric functions with respect to variables $(x_i)_{i \in \mathbb{N}}$ that become polynomial after the shift of variables $x_i \rightarrow x_i - (d+1-i)u$.

Let us denote $e_a(t_1, t_2, \dots)$ and $h_b(t_1, t_2, \dots)$ the a -th elementary symmetric function and the b -th complete symmetric function in variables $\{t_1, t_2, \dots\}$ respectively.

We will prove the following

Theorem 2.1. If an equivariant cohomology class α of Grassmannian $\text{Gr}_{g-1}(\mathcal{H})$ is represented by symmetric function $\alpha(u, x_1, \dots, x_i, \dots)$ that becomes polynomial after the shift of variables $x_i \rightarrow x_i - (g - i)u$ then

$$k^* \alpha = \alpha_g(u, x_1, \dots, x_g)$$

where we obtain α_g from α setting $x_i = (g - i)u$ for $i > g$, $u = -\psi$ and $\{-x_i : 1 \leq i \leq g\}$ are lambda-roots (in other words, $\{x_i : 1 \leq i \leq g\}$ are Chern roots of the bundle \mathbb{E}^\vee dual to the Hodge bundle \mathbb{E}).

Schubert cycles $\bar{\Sigma}_\mu$ specify equivariant cohomology classes Ω_μ^T corresponding to Okounkov-Olshanski shifted Schur functions s_μ^* . Let us recall some basic definition of the shifted Schur functions defined in [1]. The factorial Schur polynomial depending on partition μ and variables $\{z_1, \dots, z_n\}$ is given by the formula:

$$t_\mu(z_1, \dots, z_n) = \frac{\det[(z_i \downarrow \mu_j + n - j)]}{\det[(z_i \downarrow n - j)]},$$

where the symbol $(z \downarrow k)$ stands for the k -th falling factorial power of the variable z :

$$(z \downarrow k) = \begin{cases} z(z-1) \cdots (z-k+1), & k = 1, 2, \dots; \\ 1, & k = 0. \end{cases}$$

After the change of variables $z'_i = z_i - n + i$ for $1 \leq i \leq n$, we obtain the shifted Schur polynomials $s_\mu^*(z'_1, \dots, z'_n) = t_\mu(z_1, \dots, z_n)$. The shifted Schur polynomials satisfy the stability conditions $s_\mu^*(z_1, \dots, z_n, 0) = s_\mu^*(z_1, \dots, z_n)$ which allows us to define the shifted Schur functions $s_\mu^*(z_1, z_2, \dots)$ in the sequence of variables $\{z_1, z_2, \dots\}$. The stability condition expressed in terms of factorial Schur functions looks as follows $t_\mu(z_1 - l, \dots, z_n - l)$ does not depend on l if $l \geq l(\mu) - g + 1$. For more details, see [1]. It follows from the results of [8] that the equivariant Schubert class in $H_{S^1}^*(\text{Gr}_{g-1}(\mathcal{H}))$ corresponding to the partition μ is given by the formula:

$$\Omega_\mu^T = u^{|\mu|} s_\mu^*(z_1, z_2, \dots) = u^{|\mu|} t_\mu \left(\frac{x_1 - lu}{u}, \frac{x_2 - lu}{u}, \dots \right),$$

where l is a positive integer such that $l \geq l(\mu) - g + 1$, and (z_i) is the sequence of variables defined by $z_i = (x_i + (i - g)u)/u$ for all i , and $|\mu|$, the weight of a partition μ , is defined to be $\sum_i \mu_i$. Note that $x_i + (i - g)u = 0$ for all i sufficiently large in $H_{S^1}^*(\text{Gr}_{g-1}(\mathcal{H}))$ and thus the sequence of variables (z_i) defined by $z_i = (x_i + (i - g)u)/u$ makes sense in s_μ^* . Using this statement and the Theorem 2.1, we obtain

Corollary 2.1.

$$k^* \Omega_\mu^T = (-\psi)^{|\mu|} s_\mu^*(z_1, \dots, z_g) = (-\psi)^{|\mu|} t_\mu(z'_1, \dots, z'_g)$$

where $\{z_1, \dots, z_g\}$ is the set of variables defined by $z_i = (x_i - (i - g)\psi)/(-\psi)$ for $1 \leq i \leq g$ and $\{z'_1, \dots, z'_g\}$ is the set of variables defined by $z'_i = (x_i + l\psi)/(-\psi)$ for $1 \leq i \leq g$ and l is a positive integer such that $l \geq l(\mu) - g + 1$.

The factorial Schur function is an inhomogeneous symmetric function; we will represent it as a sum of homogeneous polynomials:

$$t_\mu(x_1 - l, \dots, x_n - l) = \sum t_\mu^k(x_1, \dots, x_n),$$

where $t_\mu^k(x_1, \dots, x_n)$ is a homogeneous polynomial of degree k and $l \gg 0$ (Recall that the LHS does not depend on l for large l). We can write

$$(2.3) \quad k^* \Omega_\mu^T = \sum_k (-\psi)^{|\mu| - k} t_\mu^k(x_1, \dots, x_g).$$

Shifted Schur functions form a basis in the space of all shifted symmetric functions, and therefore we can say that conversely Theorem 2.1 follows from Corollary 2.1.

Denote Ψ_μ the $l(\mu) \times l(\mu)$ matrix whose ij -th entry is given by

$$(\Psi_\mu)_{ij} = \begin{cases} \sum_{a+b=\mu_i+j-i} h_a(x_1, \dots, x_g) e_b(0, 1, 2, \dots, \mu_i - i + g - 1) \psi^b, & \text{if } \mu_i - i + g \geq 1; \\ \sum_{a+b=\mu_i+j-i} e_a(x_1, \dots, x_g) h_b(0, 1, 2, \dots, i - \mu_i - g) \psi^b, & \text{if } \mu_i - i + g \leq 0. \end{cases}$$

We can also consider another matrix (of the size $l(\mu') \times l(\mu')$) defined by

$$(\Psi'_\mu)_{ij} = \begin{cases} \sum_{a+b=\mu_i+j-i} e_a(x_1, \dots, x_g) h_b(0, 1, 2, \dots, \mu'_i - i + g - 1) \psi^b, & \text{if } \mu'_i - i + g \geq 1; \\ \sum_{a+b=\mu_i+j-i} h_a(x_1, \dots, x_g) e_b(0, 1, 2, \dots, i - \mu'_i - g) \psi^b, & \text{if } \mu'_i - i + g \leq 0. \end{cases}$$

Here μ' denotes the conjugate partition of μ . Using the determinant formula for double Schur functions, we obtain

$$k^* \Omega_\mu^T = \det \Psi_\mu = \det \Psi'_\mu.$$

If $l(\mu) \leq g$, $\mu_i - i + g > 1$ for $1 \leq i \leq g$. Thus

$$k^* \Omega_\mu^T = \det \left[\sum_{a+b=\mu_i+j-i} h_a(x_1, \dots, x_g) e_b(1, 2, \dots, \mu_i - i + g - 1) \psi^b \right]_{1 \leq i, j \leq l(\mu)}.$$

Similarly, we can also obtain the dual formula

$$k^* \Omega_\mu^T = \det \left[\sum_{a+b=\mu'_i+j-i} e_a(x_1, \dots, x_g) h_b(1, 2, \dots, \mu'_i - i + g - 1) \psi^b \right]_{1 \leq i, j \leq l(\mu')},$$

when $l(\mu) \leq g$. These two formulas are useful when we compute the cohomology classes of the Weierstrass cycles.

We can consider also cohomology classes p_s corresponding to symmetric functions

$$p_s(u, x_1, \dots, x_n, \dots) = \sum_{i=1}^{\infty} \{x_i^s - (-1)^s (i - d - 1)^s u^s\}$$

(these classes constitute a multiplicative system of generators of equivariant cohomology). Applying Theorem 2.1, we obtain

Corollary 2.2.

$$k^* p_s = \text{ch}_s(\mathbb{E}) - \sum_{i=1}^g (i - g)^s \psi^s,$$

where $\text{ch}_s(\mathbb{E})$ stands for the s -th component of the Chern character of Hodge bundle \mathbb{E} .

As we have noticed to prove Theorem 2.1, it is sufficient to prove Corollary 2.1. We will give the proof using the constructions of [9].

Let $\mathcal{H}_{i,j}$ be the linear subspace of \mathcal{H} spanned by $\{z^k : i \leq k \leq j\}$ and denote $\underline{\mathcal{H}}_{i,j}$ the product bundle $\mathcal{H}_{i,j} \times \text{Gr}_d^l$. We consider the action of S^1 on $\underline{\mathcal{H}}_{i,j}$ defined by

$$(2.4) \quad (t, (f, V)) \mapsto (t^{-1} f(t^{-1} z), t(V)).$$

Here V is a point in $\text{Gr}_d(\mathcal{H})$, f is vector in $\mathcal{H}_{i,j}$ and $t \in S^1$; Here we define $t(V)$ as the space of functions $t^{-1}f(t^{-1}z)$ for $f(z) \in V$. Then $\underline{\mathcal{H}}_{i,j}$ is an equivariant vector bundle over Gr_d^l . Then the total equivariant Chern classes of the bundle $\underline{\mathcal{H}}_{i,j}$ is given by the formula

$$c^T(\underline{\mathcal{H}}_{i,j}) = \prod_{m=i}^j (1 - (m+1)u).$$

Let f_{ln} and f_l be the inclusion maps $\text{Gr}_d^l \hookrightarrow \text{Gr}_d^n$ and $\text{Gr}_d^l \hookrightarrow \text{Gr}_d(\mathcal{H})$ respectively. The induced map of f_{ln} and f_l on the equivariant cohomology are denoted by f_{ln}^* and f_l^* respectively. The equivariant Schubert cycle $\bar{\Sigma}_\mu$ and Gr_d^l are in general position if $l(\mu) < l$. In [9], we show that

$$\Omega_{\mu,l}^T = \det [c_{\mu_i+j-i}^T(\underline{\mathcal{H}}_{-l,\mu_i-i+d-1} - \mathcal{E}_l)],$$

where $\Omega_{\mu,l}^T = f_l^* \Omega_\mu$. If we denote by x_1, \dots, x_{d+l} the equivariant Chern roots of \mathcal{E}_l^* and $y_j = (j+d+1)u$, then $\Omega_{\mu,l}^T$ is given by the double Schur function:

$$\Omega_{\mu,l}^T = {}^{d+l}s_\mu(x_1, \dots, x_{d+l} | \tau^{d+l+1}y),$$

where $\tau : \mathbb{C}[y] \rightarrow \mathbb{C}[y]$ is the translation operator $(\tau y)_i = y_{i-1}$. On the other hand, for any $n > l$, we also have $f_{ln}^* \Omega_{\mu,n}^T = \Omega_{\mu,l}^T$ if $l(\mu) < l < n$. This formula is equivalent to

$$(2.5) \quad f_{nl}^* {}^{d+n}s_\mu(x_1, \dots, x_{d+n} | \tau^{d+n+1}y) = {}^{d+l}s_\mu(x_1, \dots, x_{d+l} | \tau^{d+l+1}y).$$

Since $f_{ln}^* \mathcal{E}_n = \mathcal{E}_l \oplus \underline{\mathcal{H}}_{-n,-l-1}$, the relation (3.2) is equivalent to the following identification

$${}^{d+n}s_\mu(x_1, \dots, x_{d+n} | \tau^{d+n+1}y) = {}^{d+l}s_\mu(x_1, \dots, x_{d+l} | \tau^{d+l+1}y)$$

by requiring that $x_i = (d+1-i)u$ for $d+l+1 \leq i \leq n+d$ when $l(\mu) < l < n$. We thus obtain a double Schur function $s_\mu(x|y)$ which is equal to the equivariant Schubert class Ω_μ^T .

All statements proved above are valid not only for the space $\widehat{\mathcal{CM}}_g$, but also for its S^1 -invariant subspaces, in particular, for the subspace $\widehat{\mathcal{M}}_g$ consisting of smooth curves. For $\widehat{\mathcal{M}}_g$, some of our statements can be simplified.

For the moduli space of pointed smooth curves [15], the Mumford formula

$$(2.6) \quad c(\mathbb{E})c(\mathbb{E}^*) = 1$$

implies that $h_a(x_1, \dots, x_g) = (-1)^a \lambda_a$. Hence the Ψ -matrix can be expressed in the form:

$$(2.7) \quad (\Psi_\mu)_{ij} = \begin{cases} \sum_{a+b=\mu_i+j-i} (-1)^a e_b(0, 1, 2, \dots, \mu_i-i+g-1) \lambda_a \psi^b, & \text{if } \mu_i-i+g \geq 1; \\ \sum_{a+b=\mu_i+j-i} (-1)^a h_b(0, 1, 2, \dots, i-\mu_i-g) \lambda_a \psi^b, & \text{if } \mu_i-i+g \leq 0. \end{cases}$$

If we are working with the moduli space $\widehat{\mathcal{M}}_g$, the Chern character of the Hodge bundle can be expressed in terms of kappa-classes [15]. Therefore we obtain:

Corollary 2.3.

$$k^* p_s = \begin{cases} \sum_{i=1}^g (i-g)^{2r} \psi^{2r}, & \text{if } s = 2r; \\ B_{2r\kappa_{2r}}/2r - \sum_{i=1}^g (i-g)^{2r-1} \psi^{2r-1}, & \text{if } s = 2r-1. \end{cases}$$

3. WEIERSTRASS CYCLES

Theorem 3.1. A point $k_1(C, p, z)$ of the Krichever locus belongs to the Schubert cell Σ_S defined by the Weierstrass sequence S at the point p .

Recall that the Weierstrass sequence S is closely related to the Weierstrass semigroup H : if S' denotes the set $\{s_1 + 1, \dots, s_g + 1\}$ obtained by shifting the first g elements of the sequence S by 1 then $H = \mathbb{N} \setminus S'$.

Note that the intersection \widehat{W}_S of the Krichever locus $k_1(\widehat{\mathcal{CM}}_g)$ with the Schubert cell Σ_S can be empty; this happens, in particular, in the case when $\mathbb{N} \setminus S'$ is not a semigroup. If $\mathbb{N} \setminus S' = H$ is a semigroup then \widehat{W}_S is closely related to the variety W_H : a pair (C, p) belongs to $W_H \subset \mathcal{M}_{g,1}$ iff the triple (C, p, z) belongs to $\widehat{W}_S \subset \widehat{\mathcal{M}}_g$. Taking into account that the cohomology of $\mathcal{M}_{g,1}$ coincides with the equivariant cohomology of $\widehat{\mathcal{M}}_g$, we obtain the information about the homological class of Weierstrass cycle W_H from the properties of the homomorphism induced by the Krichever map on equivariant cohomology.

Let us consider the closure $\overline{\Sigma}_S$ of a Schubert cell Σ_S . A point $k_1(C, p, z)$ belongs to $\overline{\Sigma}_S$ if and only if the Weierstrass sequence $(s_i(p))$ at p obeys the relation $s_i(p) \geq s_i$ for all i . One can prove that for every Weierstrass sequence of genus g we have $s_k(p) \leq 2g - 2k$ ([11], Lemma 3.2.) We see that the intersection of $\overline{\Sigma}_S$ with the Krichever locus $k_1(\widehat{\mathcal{CM}}_g)$ can be non-empty only in the case when $s_k \leq 2g - 2k$ for $1 \leq k \leq g$.

If Y is an algebraic subvariety of non-singular variety X , we can use the intersection theory to study the cohomology homomorphism i^* induced by the embedding $i : Y \rightarrow X$. In particular, if a cohomology class $\nu \in H^k(X)$ is dual to a subvariety V having a codimension k in X and the intersection $V \cap Y$ is empty, we can say that $i^*\nu = 0$. If V and Y are in general position (this means that for every common point of Y and V the intersection of tangent space to V with the tangent space of Y has codimension k), then the cohomology class $i^*\nu$ is dual to the fundamental cycle of the intersection $V \cap Y$. In more general cases when we assume only that $V \cap Y$ has codimension k in Y , one can say that the class $i^*\nu$ is dual to a linear combination of irreducible components of $V \cap Y$, see [5]. (Even if $V \cap Y$ is irreducible, we can say only that $i^*\nu$ is dual to a multiple of the fundamental cycle of the intersection $V \cap Y$.)

One can apply similar statements to the case when a group G acts on X and we consider equivariant cohomology. (In this case, one should assume that Y and V are G -invariant.) Moreover, we will apply them to the case when X , Y , and V are infinite-dimensional under the assumption that V has finite codimension in X .

If the intersection \widehat{W}_S of $\overline{\Sigma}_S$ with the Krichever locus $k_1(\widehat{\mathcal{CM}}_g)$ is empty, then the homomorphism k^* determined by the Krichever map sends the equivariant cohomology class Ω_μ^T into a trivial cohomology class. Using Theorem 2.1, we obtain a relation in the tautological ring :

$$(3.1) \quad \det \Psi_\mu = 0.$$

Here μ stands for a partition corresponding to the sequence S . In particular, the above relation is satisfied if the sequence violates the relations $s_k \leq 2g - 2k$ for $1 \leq k \leq g$ and $s_k \leq k - g$ for $k \geq g + 1$. This relation can be expressed also in terms of shifted Schur functions or factorial Schur functions

$$(3.2) \quad s_\mu^*(z_1, \dots, z_g) = t_\mu \left(-\frac{x_1 + l\psi}{\psi}, \dots, -\frac{x_g + l\psi}{\psi} \right) = 0,$$

where $z_i = (x_i - (i - g)\psi)/(-\psi)$ for $1 \leq i \leq g$ and $l \geq l(\mu) - g + 1$. Probably, the most convenient way to express the relations we found is to use the functions t_μ^k (homogeneous components of factorial Schur functions) as in 2.3:

Theorem 3.2. If μ is a partition corresponding to such a sequence S that one cannot find a Weierstrass sequence S' obeying $S' \geq S$ then

$$\sum_k (-\psi)^{|\mu|-k} t_\mu^k(x_1, \dots, x_g) = 0.$$

Of course, these relations are valid also in the case when we restrict ourselves to smooth curves; we obtain relations in the tautological ring of the universal curve $\mathcal{M}_{g,1}$. Using pull-push formula we get relations in \mathcal{M}_g :

$$(3.3) \quad \sum_k (-1)^{|\mu|-k} \kappa_{|\mu|-k-1} t_\mu^k(x_1, \dots, x_g) = 0.$$

However, in the tautological rings of $\mathcal{M}_{g,1}$ and \mathcal{M}_g there exist other relations, in particular, the relations following from the Mumford formula 2.6. Notice that using (2.6) one can get on $\mathcal{M}_{g,1}$ the relation (3.1) where the Ψ -matrix is defined by (2.7).

Every point V in the Krichever locus is contained in the closed subspace \mathcal{H}' of \mathcal{H} spanned by $\{z^i : i \neq -1\}$. The space \mathcal{H}' has a natural polarized structure coming from the polarized structure of \mathcal{H} . This means that the Krichever map k sends $\widehat{\mathcal{CM}}_g$ to $\text{Gr}_g(\mathcal{H}')$. Schubert cells in $\text{Gr}_g(\mathcal{H}')$ are labeled by sequences S obeying $s_i = g - 1 - i$ for $i \gg 0$; we will use the notation Σ'_S for these cells. It is easy to check that $\Sigma'_S = \Sigma_S \cap \text{Gr}_g(\mathcal{H}')$. Assume that $k^{-1}\Sigma'_S$ is nonempty. Then a point $(C, P, z) \in k^{-1}\Sigma'_S$ if and only if P has Weierstrass sequence S . If $\overline{\Sigma}'_S$ and the Krichever locus $k_1(\widehat{\mathcal{CM}}_g)$ are in general position in $\text{Gr}_g(\mathcal{H}')$, the codimension of the intersection of the Weierstrass cycle $\widehat{W}_S = \overline{\Sigma}'_S \cap k_1(\widehat{\mathcal{CM}}_g)$ in $k_1(\widehat{\mathcal{CM}}_g)$ is equal to the codimension of $\overline{\Sigma}'_S$ in $\text{Gr}_g(\mathcal{H}')$ ²; we can say that the equivariant cohomology class $[W_S]$ corresponding to \widehat{W}_S is equal to $k^*\Omega_\mu^T$, where Ω_μ^T is the equivariant cohomology class corresponding to the Schubert cycle $\overline{\Sigma}'_S$ in the equivariant cohomology of Grassmannian $\text{Gr}(\mathcal{H}')$. Let us impose a weaker condition that the codimension of intersection is equal to codimension of $\overline{\Sigma}'_S$ in $\text{Gr}(\mathcal{H}')$ and assume that the intersection is irreducible. In this situation we will say that the Weierstrass cycle W_S is generic. For generic Weierstrass cycle the corresponding cohomology class $[W_S]$ is equal to $k^*\Omega_\mu^T$ up to a constant factor.³ If the condition on the codimension of intersection is satisfied, but the intersection is reducible $k^*\Omega_\mu^T$ is equal to linear combination of cohomology classes corresponding to irreducible components of W_S .

Let $s_\mu^*(z_1, z_2, \dots)$ be the shifted Schur function of partition μ in variables z_1, z_2, \dots defined in [1]. Using the calculation of $k^*\Omega_\mu^T$ in Section 2, we can obtain information about $[W_S]$: If $\overline{\Sigma}'_S$ and $k_1(\widehat{\mathcal{CM}}_g)$ are in general position, then

$$(3.4) \quad [W_S] = (-\psi)^{|\mu|} s_\mu^*(z_1, \dots, z_g) = (-\psi)^{|\mu|} t_\mu \left(-\frac{x_1}{\psi}, \dots, -\frac{x_g}{\psi} \right),$$

²The complex codimension of $\overline{\Sigma}'_S$ in $\text{Gr}_g(\mathcal{H}')$ is equal to $|S| = \sum_{i=1}^{i_0} (s_i + i - g) + \sum_{i=i_0+1}^{\infty} (s_i + i - g + 1)$, where i_0 is the index so that $s_{i_0} \geq 0$ and $s_{i_0+1} < 0$. If $s_i < 0$ for all i , we set $i_0 = 0$.

³Our derivation is based on the consideration of intersections of infinite-dimensional varieties. It is not very easy to make it rigorous. A rigorous proof can be based on the Porteous formula or its generalizations.

where μ is the partition corresponding to the sequence S and z_1, \dots, z_g are the formal variables defined by $z_i = (x_i - (i - g - 1)\psi)/(-\psi)$ for $1 \leq i \leq g$.⁴

Again it is more convenient to use homogeneous components of factorial Schur functions

Theorem 3.3. If $\overline{\Sigma}'_S$ and $k_1(\widehat{\mathcal{CM}}_g)$ are in general position, then

$$(3.5) \quad [W_S] = \sum_k (-\psi)^{|\mu| - k} t_\mu^k(x_1, \dots, x_g)$$

where μ stands for the partition corresponding to S . If H is a Weierstrass semigroup and S is the corresponding Weierstrass sequence, then (3.5) gives us the cohomology class $[W_H]$ of W_H . Assume that we impose a weaker condition that W_S is generic. Then the formula (3.5) is true up to a constant factor⁵.

Notice that in the case when the codimension of \widehat{W}_S is not equal to codimension of $\overline{\Sigma}'_S$ in $\text{Gr}_g(\mathcal{H}')$ the RHS of (3.4) makes sense, but is not related to the homology of Weierstrass cycle. One can say that it specifies the cohomology of “virtual” Weierstrass cycle. It is interesting to notice that the multiplication rule of Schubert classes in equivariant cohomology of Grassmannian (see [9],[13]) gives a multiplication rule for “virtual” W cycles.

One can consider Weierstrass cycles in \mathcal{M}_g defined as images of Weierstrass cycles in $\mathcal{M}_{g,1}$ by the forgetful map. In other words, we define W'_S as a subvariety consisting of curves $C \in \mathcal{M}_g$ containing at least one point with Weierstrass sequence S . Using the pull-push formula, we obtain the following expression for corresponding cohomology classes

$$(3.6) \quad [W'_S] = \text{const} \sum_k (-1)^{|\mu| - k} \kappa_{|\mu| - k - 1} t_\mu^k(x_1, \dots, x_g).$$

Here μ stands for the partition corresponding S and $\kappa_b = \pi_* \psi^{b+1}$ are the kappa-classes. This expression is valid if W'_S has the expected codimension in \mathcal{M}_g , i.e. the expression holds if the codimension of W'_S in \mathcal{M}_g equals to $2|S| - 2$.

In a separate paper [10], we will apply the results of present paper to low genera. We estimate the dimension of Weierstrass cycles from below; using the calculations of [11] and [12] we show that for $g \leq 6$ this estimate either coincides with exact dimension or differs by one. If our estimate coincides with exact dimension we are able to calculate the homology class of Weierstrass cycle up to a constant factor; we performed this calculation for $g \leq 6$. We compare the relations in the tautological ring obtained in present paper with the description of the tautological ring of \mathcal{M}_g obtained by Faber [3].

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⁴One can show that the partition corresponding to a Weierstrass sequence has length at most g . Therefore the factorial Schur function $t_\mu(x_1 - l, \dots, x_g - l)$ is already in stable range for $l = 0$.

⁵It follows from Serre’s theorem [16] that this factor does not vanish

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